WEST BENGAL STATE UNIVERSITY

B.Sc. Honours Part-III Examination, 2020

MATHEMATICS

PAPER-MTMA-VIII-A

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

GROUP-A

SECTION-I

(LINEAR ALGEBRA)

Answer any *one* question from the following $10\times 1 = 10$

1. (a) Prove that two finite dimensional vector spaces V and W over a field F are isomorphic if and only if $\dim V = \dim W$. 4

   (b) Let $P_2(\mathbb{R})$ be the vector space of polynomials in x of degree atmost 2 with real coefficients and $M_2(\mathbb{R})$ be the vector space of 2×2 real matrices. Write the standard order bases B_1 of $P_2(\mathbb{R})$ and B_2 of $M_2(\mathbb{R})$. Determine the matrix $[T]$ of the linear transformation $T: P_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ defined by $T(f) = \begin{bmatrix} f(0)-f(2) & 0 \\ 0 & f(1) \end{bmatrix}$, relative to the pair of ordered bases B_1 and B_2. Hence, out of the matrix $[T]$, find the rank and nullity of T. Also, determine a basis of range space of T. 1+2+2+1

2. (a) For a positive integer n, P_n denotes the vector space of polynomials of degree $\le n$, over the field of real numbers. Let $T: P_2 \to P_4$ be a linear transformation defined by $T(f(x)) = 2f'(x) + \int_0^x t\, f(t)\, dt$, for all $f(x) \in P_2$. Prove that T is injective. 4

   (b) Let $B_1 = \{(1, 2), (2, -1)\}$, $B_2 = \{(1, 0), (0, 1)\}$ be two ordered bases of $\mathbb{R}^2$. If the matrix of a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ relative to the pair of ordered bases B_1 and B_2 is given by $\begin{bmatrix} 4 & 3 \\ 2 & -4 \end{bmatrix}$, find the vector $T(5, 5)$ in $\mathbb{R}^2$. 3

   (c) Let $T_1, T_2: \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be linear transformations such that $\operatorname{rank}(T_1) = 3$ and nullity $(T_2) = 3$. Let $T_3: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $T_3 \circ T_1 = T_2$. Find the rank of T_3. ($T_3 \circ T_1$ stands for mapping composition of T_3 and T_1). 3

SECTION-II

(MODERN ALGEBRA)

Answer any *one* question from the following $8\times1=8$

3. (a) Let $f:G\to G'$ be a homomorphism of groups. Prove that $\ker f$ is a normal subgroup of *G*. Also prove that f is injective if and only if $\ker f=\{e_G\}$, where e_G represents the identity element of *G*. 4

(b) Show that the additive group of rational numbers *Q* is not isomorphic to the multiplicative group of positive rational numbers Q^+. 2

(c) Let *G* be a group and $f:G\to G$ be a function defined by $f(a)=a^{-1}$ for any $a\in G$. Prove that f is a homomorphism if and only if *G* is commutative. 2

4. (a) If *H* and *K* are normal subgroups of a group *G* such that $H\cap K=\{e_G\}$ where e_G is the identity element of *G*, then show that $hk=kh$ for all $h\in H$ and $k\in K$. 2

(b) If *H* is the only subgroup of order *n* in a group *G*, then prove that *H* is a normal subgroup of *G*. 2

(c) If *H* is a subgroup of a group *G* and $N(H)=\{x\in G:\, xHx^{-1}=H\}$, then show that 2+2

(i) $N(H)$ is a subgroup of *G* and

(ii) *H* is normal in $N(H)$.

SECTION-III

(BOOLEAN ALGEBRA)

Answer any *one* question from the following $7\times1=7$

5. (a) Prove that the set *S* of all positive divisors of 70 forms a Boolean algebra $(S,\vee,\wedge,')$, where 4

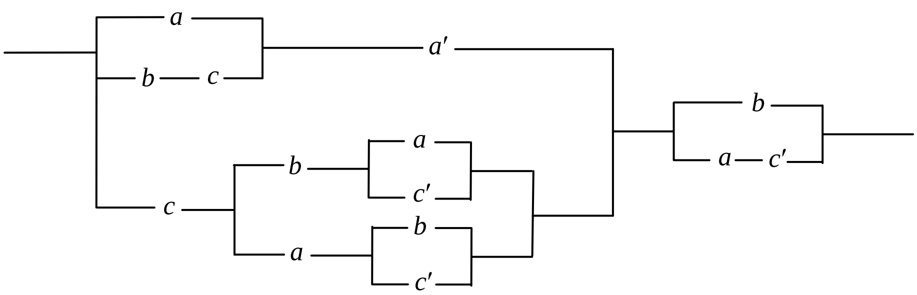
$$a\vee b=\text{l.c.m. of } a, b$$

$$a\wedge b=\text{g.c.d. of } a, b$$

$$a'=\frac{70}{a}$$

for any $a, b\in S$.

(b) Find the Boolean function that represents the circuit below and hence find the simplest circuit. 3



6. (a) In a Boolean algebra, show that 2+2

(i) $a + a'b = a + b$

(ii) $(a+b')(a'+b')(a+b)(a'+b) = 0$

for any $a, b \in B$.

(b) Express the Boolean expression $(x+y)(x+y')(x'+z)$ in DNF in the variables x, y and also express it in DNF in the variables x, y, z. 3

GROUP-B

(DIFFERENTIAL EQUATION-III)

Answer any *one* question from the following 15×1 = 15

7. (a) Find the power series solution of $y'' - xy' + x^2 y = 0$ about $x = 0$. 5

(b) Find the inverse Laplace transform of $\dfrac{1}{s^3(s^2+1)}$. 5

(c) If $L^{-1}\{f(s)\} = F(t)$, then prove $L^{-1}\{e^{-as} f(s)\} = G(t)$, where 5

$$G(t) = \begin{cases} F(t-a) & , \ t > a \\ 0 & , \ t < a \end{cases}$$

where L^{-1} denotes inverse Laplace transform.

8. (a) Find power series solution of $y'' + (x-1)y' + y = 0$ in power of $(x-2)$. 5

(b) Use the convolution theorem to find $L^{-1}\left\{\dfrac{1}{(s-1)\sqrt{s}}\right\}$, where L^{-1} denotes inverse Laplace transform. 5

(c) Solve using Laplace transform: $\dfrac{d^2y}{dx^2} + 9y = \cos 2t$ if $y(0) = 1$, $y(\pi/2) = -1$. 5

GROUP-C

(TENSOR CALCULUS)

Answer any *one* question from the following 10×1 = 10

9. (a) Let a_{ij} be a symmetric (0, 2) type tensor satisfying $|a_{ij}| \neq 0$ and let b^{ij} be the cofactor of a_{ij} in $|a_{ij}|$ divided by $|a_{ij}|$. Prove that b^{ij} is a symmetric (2, 0) type tensor. 3

(b) A^{lm}_{ij} is a (2, 2) type tensor and B^i_j is a (1, 1) type tensor. Prove that $A^{lm}_{ij} B^i_j$ is (2, 2) type tensor. 2

(c) Define covariant derivative $A_{ij,k}$ of a (0, 2) type tensor A_{ij} and prove that $A_{ij,k}$ is symmetric in *i* and *j* if A_{ij} is symmetric. Find the value of $g_{ij,k}$, where g_{ij} is the fundamental tensor. 1+2+2

10.(a) The components of a covariant tensor in the *x*-system are $A_{11} = 4,\ A_{12} = A_{21} = 0,\ A_{22} = 7$. Find its components in the $\bar{x}$-system where 3

$$x^1 = 4(\bar{x}^1)^2 - 7(\bar{x}^2)^2$$
$$x^2 = 4\bar{x}^1 - 5\bar{x}^2$$

(b) Show that in an *n*-dimensional space a covariant skew-symmetric tensor of second order has at most $\frac{1}{2}n(n-1)$ different arithmetic components. 3

(c) Show that in a Riemannian space V_n of dimension *n* with metric tensor g_{ij}, 4

$$\begin{Bmatrix} i \\ i \quad j \end{Bmatrix} = \frac{\partial}{\partial x^j}(\log\sqrt{g})$$

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