WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2020

MTMACOR10T-MATHEMATICS (CC10)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Show that the characteristic of a ring R with unity 1 is $n(>0)$ if and only if $n.1=0$.

(b) Let R be a ring with $a^2=a$ for all $a\in R$. Prove that $a+b=0 \Rightarrow a=b$.

(c) Let S be a nonempty subset of a ring R. Show that S is a subring of R if and only if $\forall\ x, y\in S, x-y\in S$ and $x.y\in S$.

(d) If F is a field, then show that F has no non-trivial ideal.

(e) Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

(f) If W_1, W_2 are two subspaces of a vector space V over a field F such that $W_1+W_2=V$ and $W_1\cap W_2=\{0\}$ then prove that for each vector $\alpha\in V$ there are unique vectors $\alpha_1\in W_1$ and $\alpha_2\in W_2$ such that $\alpha=\alpha_1+\alpha_2$.

(g) Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, γ are linearly independent vectors of V. Prove that $(\alpha+\beta)$, $(\beta+\gamma)$ and $(\gamma+\alpha)$ are linearly independent.

(h) Let V and W be two vector spaces over the same field F and let $T:V\to W$ be a linear transformation. If V is finite dimensional, define the rank and nullity of T.

2. (a) Prove that a commutative ring R satisfies cancellation property for multiplication if and only if R has no zero divisors. 4

(b) Prove that the characteristic of an integral domain is either zero or a prime integer. 4

3. (a) Show that the set of integers modulo 6 form a ring with respect to the addition and multiplication modulo 6. 3+1

Is it an integral domain? — Justify your answer.

(b) Prove that every finite integral domain is a field. Give an example to show that the result is false if the 'finiteness' condition is dropped. 3+1

4. (a) Let R be a commutative ring with identity 1. Show that an ideal M in R is maximal if and only if the quotient ring R/M is a field. 4

(b) Let I be an ideal of a commutative ring R. Define a subset S of R by $S=\{r\in R : ra=0 \text{ for all } a\in I\}$. Prove that S is an ideal of R. 4

5. (a) Let f be a homomorphism of a ring R into a ring R'. Show that $f(R)$ is an ideal of R' and $R/\ker f \simeq f(R)$. 1+3

(b) Show that $\mathbb{Z}_n$, the ring of integers modulo n and the quotient ring $\mathbb{Z}/\langle n\rangle$ are isomorphic, where $\langle n\rangle=\{m\in\mathbb{Z} : m=qn \text{ for some } q\in\mathbb{Z}\}$. 4

6. (a) Show that the mapping $f:\mathbb{Z}_6\to\mathbb{Z}_{10}$ defined by $f([a])=5[a]$ for all $[a]\in\mathbb{Z}_6$ is a ring homomorphism from the ring $\mathbb{Z}_6$ into the ring $\mathbb{Z}_{10}$. 4

(b) Define Kernel of a ring homomorphism $f: R\to S$ from a ring R into a ring S. Prove that $\ker f$ is an ideal of R. 4

7. (a) Prove that every set of linearly independent vectors of a finite dimensional vector space is either a basis or can be extended to a basis of the vector space. 3

(b) Let $W=\{(x,y,z)\in\mathbb{R}^3 : x-4y+3z=0\}$. Show that W is a subspace of $\mathbb{R}^3$. Also find a basis of W. 2+3

8. (a) Let V and W be two vector spaces over a field F. Prove that a necessary and sufficient condition for a linear mapping $T: V\to W$ to be invertible is that T is one-to-one and onto. 4

(b) A linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ is defined by 4

$$T(x_1,\ x_2,\ x_3)=(2x_1+x_2-x_3,\ \ x_2+4x_3,\ x_1-x_2+3x_3),\ \ (x_1,\ x_2,\ x_3)\in\mathbb{R}^3$$

Find the matrix representation of T relative to the ordered basis (0, 1, 1), (1, 0, 1), (1, 1, 0) of $\mathbb{R}^3$.

9. (a) If V and W be two finite dimensional vector spaces and $T: V\to W$ is a linear transformation, then show that dim V = nullity of T + rank of T. 4

(b) Find the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$, if 2+2

T(1, 0, 0) = (2, 3, 4), T(0, 1, 0) = (1, 5, 6) and T(1, 1, 1) = (7, 8, 4).

Also find its matrix representation with respect to {(1, 0, 0), (0, 1, 0), (1, 1, 1)}.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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